Physics 402
Prof. Anlage
Discussion Worksheet

WKB Approximation applied to the Harmonic Oscillator. Find the eigen-energies of the bound states of a 1D harmonic oscillator ($V(x) = m\omega^2 x^2/2$) using the WKB approximation.

Hint 1: WKB for finite well with classical turning points at x_1 and x_2:
$\int_{x_1}^{x_2} \sqrt{2m(E - V(x'))}\, dx' = \pi\hbar \left(n - \frac{1}{2}\right)$ with $n = 1, 2, 3, \ldots$.

Hint 2: Total energy $E = \frac{1}{2}\, m\, \omega^2 A^2$, where $\pm A$ are the classical turning points.

Hint 3: $\int_{-\pi/2}^{+\pi/2} cos^2\theta \; d\theta = \frac{1}{2}\,\pi$

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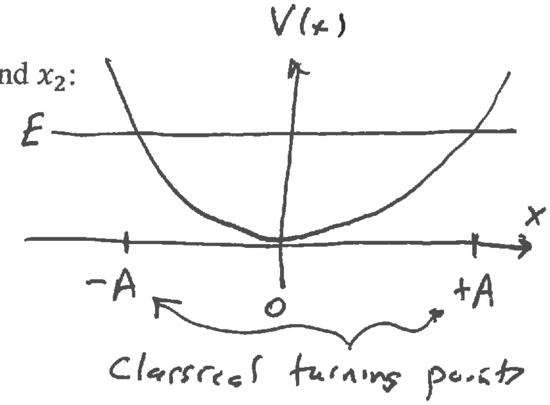

Hint 1: WKB for finite well with classical turning points at x_1 and x_2: $\int_{x_1}^{x_2} \sqrt{2m(E - V(x'))}\, dx' = \pi\hbar\left(n - \frac{1}{2}\right)$ with $n = 1, 2, 3, \ldots$

Hint 2: $\int_{-\pi/2}^{+\pi/2} \cos^2\theta \, d\theta = \frac{1}{2}\pi$

$$\int_{-A}^{+A} \sqrt{2m\left(E - m\omega^2 (x')^2/2\right)}\, dx' = \pi\hbar\left(n - \tfrac{1}{2}\right)$$

Write the total energy as $E = \frac{1}{2}m\omega^2 A^2$, so the integral becomes

$$I = \sqrt{2m}\sqrt{\frac{m\omega^2}{2}} \int_{-A}^{A} \sqrt{A^2 - (x')^2}\, dx'$$

Now substitute $x' = A\sin\theta$
$dx' = A\cos\theta\, d\theta$
$x = A \Rightarrow \theta = +\pi/2$
$x = -A \Rightarrow \theta = -\pi/2$

So the integral becomes

$$I = m\omega \int_{-\pi/2}^{\pi/2} A\cos\theta\, A\cos\theta\, d\theta = m\omega A^2 \underbrace{\int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta}$$

The average value of $\cos^2\theta$ over an integer number of "loops" is $1/2$. This integral is just $\frac{1}{2} \times \pi$.

$$I = m\omega A^2 \frac{\pi}{2} = \frac{E\pi}{\omega^2}\omega = \frac{E}{\omega}\pi$$

So from the WKB equation, we have

$$\cancel{\pi}\frac{E}{\omega} = \cancel{\pi}\hbar\left(n - \tfrac{1}{2}\right) \quad \text{or} \quad E = \hbar\omega\left(n - \tfrac{1}{2}\right) \quad n = 1, 2, 3, \ldots$$

or finally, $E = \left(p + \frac{1}{2}\right)\hbar\omega$, $p = 0, 1, 2, 3, \ldots$

The exact eigen-energies of the harmonic oscillator!